Positive Tail Loads for Minimum Induced Drag of Subsonic Aircraft

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By applying Prandtl's relation for the induced drag of a biplane to typical wing-tail combinations, it can be shown that the minimum induced drag occurs with a positive tail upload. This fact has been overlooked because the reduction in the total induced drag by a tail download was overestimated by using the total downwash of the wing on the tail, while neglecting the downwash produced on the wing by the tail. It is proved that, regardless of the relative size of the tail, the downwash produced by a tail download increases the induced drag of the wing so as to cancel the additional "tail thrust," and keep the mutually induced drag of a wing-tail combination the same as that induced upon the tail alone when it is in the wing's Trefftz-plane. At any finite tail length the bound circulation vortex of the wing produces a downwash that increases the induced drag of a tail upload. However, the circulation vortex system of the tail upload produces an upwash on the wing that results in a "wing thrust" component that cancels the increased drag on the tail so that the total induced drag is a minimum with a positive tail load. In order to facilitate the calculation of the mutually induced drag of typical wing-tail combinations, an explicit relation is derived for the limiting case of a small-span tail at any distance above or below a large-span wing.

Nomenclature

A	$=b^2/S=$ aspect ratio of wing (1) or tail (2)
b	= span of wing (b_1) or tail (b_2)
C_1	= mean aerodynamic chord of the wing
C_D	$=D/q\,S_1$
C_L	$=L/q\,S_1$
C_m	$=M/q\,S_1C_1$
D	= aerodynamic drag force
E	$=(q_2/q)(S_2/S_1)=\eta(S_2/S_1)$
e_1	= induced drag correction for the wing
g	= gap or vertical separation of wing and tail
h_0C_1	= distance of tail-off neutral point from leading edge of C_1
L	= aerodynamic lift force
l	= tail length or stagger distance
ΔM_α	= unstable aerodynamic pitching moment of the fuselage
q	$=\frac{1}{2}\rho v^2=$ freestream dynamic pressure for the wing
S	= lifting surface planform area
v	= freestream velocity for the wing
$w_1(x,y,z)$	= total downwash velocity produced by the wing
W	$=L_1+L_2=$ aircraft's total weight
α	= angle of attack of the wing
ϵ_W	$=w_1/v=$ total downwash angle produced by the wing
η	$=q_2/q=$ tail efficiency
ρ	= atmospheric mass density
σ	= Prandtl's coefficient for mutually induced drag between two lifting surfaces

Subscripts

1	= wing
2	= tail
i	= induced drag
m	= minimum induced drag
0	= zero tail load
∞	= Trefftz-plane far behind the wing
12	= downwash or induced drag produced by wing upon tail

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I. Introduction

FOR ideal potential flow Naylor[1] showed that if the tail of a normal monoplane aircraft is in the same horizontal plane as the wing, then the minimum induced drag occurs when the tail load is zero. Then Laitone[2] showed that if the tail is either above or below the plane of the wing the minimum induced drag is obtained with a positive tail load. Many aircraft carry a tail download of nearly 5% of the gross weight, and if their tail volume were sufficient to maintain a stable trim condition with a zero tail load, then it will be shown that their induced drag would be decreased about 8% and their total drag decreased by over 3%.

The fact that the total induced drag of a wing-tail combination is a minimum with a positive tail upload has not been recognized in the past because the reduction in total induced drag from the "tail thrust" produced by the wing's downwash acting on a tail download has been incorrectly calculated by using the total downwash produced by the wing on the tail, while neglecting the additional drag induced on the wing by a tail download. It is proved that regardless of the relative dimensions of the tail with respect to the wing, one cannot neglect this effect of the tail upon the wing, and if the upwash produced on the wing by a tail upload is considered, then the total induced drag of the wing-tail combination is a minimum with a tail upload.

It will be shown that all real viscous flow effects will increase the positive tail upload required for the minimum induced drag, and as a result, the induced drag with a tail download is always greater than that with either a zero, or corresponding tail upload. The largest inviscid effect is found to be the decrease of the wing's downwash on a small-span tail due to the rollup of the wings trailing vortex sheet.

II. Analysis of Potential Flow Induced Drag

Naylor[1] showed that as long as the tail's vertical distance (gap) relative to the wing is very small, then the minimum induced drag occurs with a zero tail load. His analysis was based on Prandtl's[3,4] relation for the induced drag of a biplane, since the tail may be considered as the smaller wing forming a biplane with large stagger (distance of the tail behind the wing), and small gap (vertical distance of the tail above or below the wing), so that the total potential flow induced drag is given by

$$\pi q D_i = \frac{L_1^2}{b_1^2} + 2\sigma \frac{L_1}{b_1}\,\frac{L_2}{b_2} + \frac{L_2^2}{b_2^2} \tag{1}$$

where L_1 is the lift force acting on the wing of span b_1, and L_2 is the lift force acting on the tail of span $b_2 \leq b_1$. The factor σ is the Prandtl coefficient[3,4] that is dependent upon the span ratio $b_2/b_1 \leq 1$, and the gap (g = vertical distance between the wing and the tail), but is independent of the stagger distance in potential flow. Prandtl calculated σ as the mutual downwash effects on the total induced drag for elliptic load distributions with no rolling up of the trailing circulation vortices. His numerical values for σ are given in Refs. 3 – 6 in terms of the gap to average span ratio $2g/(b_1+b_2)$. Similarly calculated values for σ are given in Table 1 in terms of the gap-to-wingspan ratio (g/b_1) with some additional values added to facilitate the calculation of the total induced drag for typical wing-tail combinations.

Naylor[1] only considered the case of zero gap having $\sigma = b_2/b_1$ so that Eq. (1) reduced to

$$\pi q D_i = \frac{L_1^2}{b_1^2} + 2\frac{L_1 L_2}{b_1^2} + \frac{L_2^2}{b_2^2} = \frac{W^2}{b_1^2} + \frac{L_2^2}{b_2^2}\left(1 - \frac{b_2^2}{b_1^2}\right) \tag{2}$$

where the aircraft weight is given by $W = L_1 + L_2$ in steady level flight. This confirms Naylor's results for $b_2 < b_1$ since the minimum induced drag can occur only when $L_2 = 0$, i.e., a zero tail load.

For a finite vertical gap we have $\sigma < b_2/b_1$, and Laitone[2] showed that by introducing the relation $L_1 = W - L_2$, Eq. (1) could be written as

$$\pi q D_i = \frac{W^2}{b_1^2} - 2\left(1 - \sigma\frac{b_1}{b_2}\right)\left(\frac{W L_2}{b_1^2}\right) + \left(1 - 2\sigma\frac{b_1}{b_2} + \frac{b_1^2}{b_2^2}\right)\left(\frac{L_2}{b_1}\right)^2 \tag{3}$$

Since $\sigma < b_2/b_1 < 1$ the middle term proves that $L_2 < 0$ produces a greater induced drag than does $L_2 > 0$ (tail upload), and that the minimum total induced drag occurs when the tail upload is given by

$$\frac{L_2}{W} = \left(1 - \sigma\frac{b_1}{b_2}\right)\left(1 - 2\sigma\frac{b_1}{b_2} + \frac{b_1^2}{b_2^2}\right)^{-1} > 0 \tag{4}$$

or

$$1 > \frac{L_2}{L_1} = \frac{b_2/b_1 - \sigma}{b_1/b_2 - \sigma} > 0 \tag{5}$$

Upon substituting Eq. (4) into Eq. (3) we obtain the minimum total induced drag for $L_2 > 0$ as

$$\pi q D_m = \frac{W^2}{b_1^2}\left[1 - \left(1 - \frac{\sigma b_1}{b_2}\right)^2 \left(1 - \frac{2\sigma b_1}{b_2} + \frac{b_1^2}{b_2^2}\right)^{-1}\right]$$

$$= W^2 (1 - \sigma^2)(b_1^2 - 2\sigma b_1 b_2 + b_2^2)^{-1} \tag{6}$$

The latter expression and Eq. (5) are given on p. 184 of Glauert.[5] Figure 1 shows the variation of the minimum total induced drag as (D_m/D_0) where D_0 is the induced drag with zero tail load ($L_2 = 0$) as defined by

$$D_0 = \frac{1}{\pi q}\,\frac{W^2}{b_1^2} = \frac{C_{L0}^2}{\pi A_1} q S_1 \tag{7}$$

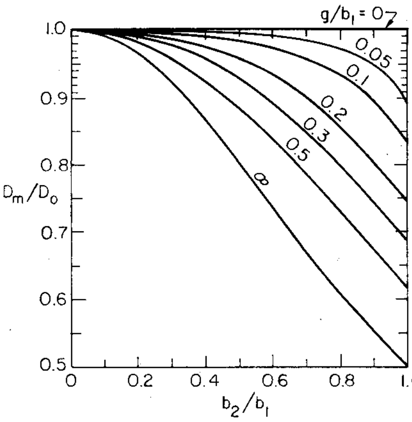

Fig. 1 Variation of minimum induced drag with tail span and gap ratio from Eq. (6).

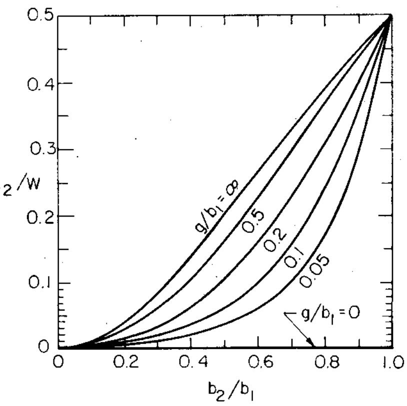

Fig. 2 Variation of tail upload for minimum induced drag from Eq. (4).

Figure 2 shows that a typical high tail on an aircraft could support an appreciable part of the total weight at the ratio given by Eq. (4) for minimum induced drag. However, the numerical values for a typical wing-tail combination with $b_2/b_1 = 0.3$ in Table 1 show that for usual tail heights the decrease in induced drag when $L_2 > 0$ is less than 2% of the corresponding drag for zero tail loads, consequently a suitable design goal would be a zero tail load. Whenever the tail is above or below the wing a tail download should be avoided because, as will be shown, the induced drag can be increased more than 8% over that for a zero tail load if $L_2/W = -0.05$ and $g/b_1 = 0.2$. This relatively large increase in the total induced drag due to a tail download seems to have been overlooked because of a common error that introduced a reduction in total induced drag by the "tail thrust." It is true that the downwash of the wing rotates the tail force vector so that a tail upload increases the drag of the tail, whereas a tail download decreases the induced drag component of the tail. However, what has been overlooked is the fact that the circulation vortex system produced by a download on a tail will rotate the wings lift vector so as to create an additional induced drag that cancels the "tail thrust." Conversely, the circulation vortex system of an upload on the tail will rotate the wings lift vector so as to produce a "wing thrust" component. This can be shown by introducing the following into Eq. (1):

$$D_i = C_{D_i} q S_1 \quad L_1 = C_{L_1} q S_1 \quad L_2 = C_{L_2} q_2 S_2 = E C_{L_2} q S_1$$

$$A_{1,2} = (b^2/S)_{1,2} \qquad E = (S_2/S_1)(q_2/q) = \eta S_2/S_1 \tag{8}$$

Then we obtain the nondimensional form of Eq. (1) as

$$C_{D_i} = \frac{C_{L_1}^2}{\pi A_1} + \frac{\sigma b_1}{b_2}\,\frac{2 C_{L_1}}{\pi A_1} E C_{L_2} + \frac{S_1}{S_2}\,\frac{(E C_{L_2})^2}{\pi A_2} \tag{9}$$

and now we eliminate C_{L_l} by introducing

$$C_{L0}=w/qS_l=C_{L_l}+EC_{L_2}$$

so as to obtain

$$C_{D_i}=\frac{C_{L0}^2}{\pi A_l}-\left(1-\frac{\sigma b_l}{b_2}\right)\frac{2C_{L0}}{\pi A_l}EC_{L_2}+\left[1+\left(\frac{b_2}{b_l}\right)^2\left(1-\frac{2\sigma b_l}{b_2}\right)\right]\left(\frac{q_2}{q}\right)^2\frac{S_2}{S_l}\frac{C_{L_2}^2}{\pi A_2} \qquad (10)$$

The variation of C_{D_i} with C_{L_2} for gap ratios of 0, 0.1, and 0.2 is shown in Fig. 3 for a typical subsonic transport aircraft. It is evident that a finite gap gives a minimum total induced drag when the tail load is positive, and in this case the induced drag is always greater with a tail download than it is with a zero tail load. For example, if $g/b_l=0.2$, then Fig. 3 shows that a tail download equal to 5% of the weight of the aircraft ($L_2/w=-0.05$, $EC_{L_2}=-0.022$) gives $C_{D_i}=0.00937$, which is 6.5% greater than $C_{D_i}=0.0088$ for a zero tail load, and 8.0% greater than $C_{D_i}=0.00868$ for the optimum tail upload ($L_2/w=0.035$, $EC_{L_2}=0.0154$). The gap ratios used in Fig. 3 represent a range used in modern subsonic transport aircraft. The Boeing 707 aircraft all have $g/b_l>0.05$, and the Boeing 727 aircraft all have $g/b_l>0.2$.

The tail thrust produced by a tail download ($C_{L_2}<0$) is given by the middle term on the right-hand side of Eq. (9), and in this form it is obvious that it is related to the wing's downwash in the Trefftz-plane, i.e., the downwash $\epsilon_{l\infty}=2C_{L_l}/\pi A_l$ produced far behind the wing by its trailing circulation vortices, and therefore does not include the downwash produced on the tail by the wing's bound circulation vortex system. Consequently, one cannot replace the middle term in Eq. (9) by $\epsilon_W EC_{L_2}$, as is usually done (e.g., see Refs. 7–9), since the finite distance of the tail behind the wing can result in

$$\epsilon_W>\epsilon_{l\infty}=\frac{2C_{L_l}}{\pi A_l}\geq\frac{\sigma b_l}{b_2}\frac{2C_{L_l}}{\pi A_l}=\frac{\sigma b_l}{b_2}\epsilon_{l\infty} \qquad (11)$$

It is easily shown that in potential flow, with no rolling up of the wings trailing vortex sheet, ϵ_W can be considerably greater than $\epsilon_{l\infty}$, e.g., see Glauert.[5] In addition, the wing downwash acting on the tail in Eq. (9) is greatly decreased by any vertical distance of the tail above or below the wing, since as shown in Table 1, $\sigma b_l/b_2\leq 1$. For example, if $b_2/b_l=0.3$ and $g/b_l=0.2$, then the value of $\sigma b_l/b_2$ is reduced 38% from its zero gap value of unity to 0.62.

The term $(\sigma b_l/b_2)(2C_{L_l}/\pi A_l)$, which appears in Eqs. (9) and (11), represents the wing's downwash in the Trefftz-plane for potential flow with an elliptic loading on the upstream lifting surface (b_l) corresponding to the wing. Then the downstream lifting surface (b_2) corresponds to the tail in the limiting case of infinite stagger. Consequently, in this case,

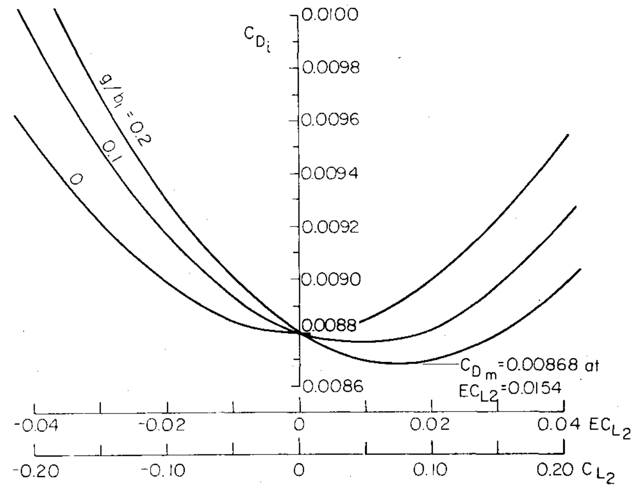

Fig. 3 Variation of C_{D_i} with C_{L_2} from Eq. (10) for various gap ratios of an aircraft defined by $e_l=e_2=e_3=1$; $b_2/b_l=0.3$, $C_{L0}=0.44$, $\pi A_l=22$, $\pi A_2=9.9$, $E=0.2$, $C_{D0}=0.0088$.

the drag induced on the wing by the tail is zero ($D_{2l}=0$ in Prandtl's[3,4] notation). Therefore, the total mutually induced drag is entirely on the tail, and given by the middle term in Eq. (9) as

$$(D_{l2})_\infty=\frac{\sigma b_l}{b_2}\frac{2C_{L_l}}{\pi A_l}EC_{L_2}qS_l=\frac{\sigma b_l}{b_2}\epsilon_{l\infty}L_2$$
$$=[\bar{w}_{l2}(\infty,0,g)/v]L_2=D_{l2}+D_{2l}=\text{constant} \qquad (12)$$

By Munk's stagger theorem, the total mutually induced drag is constant for all stagger distances, but the mutually induced drag on either the wing or the tail varies with the tail length (l = stagger distance). The distribution varies from $D_{l2}=(D_{l2})_\infty$ on the tail and $D_{2l}=0$ on the wing for large stagger distances with any gap, to $D_{l2}=\frac{1}{2}(D_{l2})_\infty=D_{2l}$ for zero stagger with a finite gap. Consequently, the distribution of the mutual drag due to the wing's trailing vortex system alone would vary with the stagger distance l in the following manner:

$$\Delta D_{l2}=\tfrac{1}{2}(D_{l2})_\infty+[\bar{w}_{l2}(l,0,g)/v-\tfrac{1}{2}\epsilon_{l\infty}(\sigma b_l/b_2)]L_2 \qquad (13a)$$

$$\Delta D_{2l}=\tfrac{1}{2}(D_{l2})_\infty-[\bar{w}_{l2}(l,0,g)/v-\tfrac{1}{2}\epsilon_{l\infty}(\sigma b_l/b_2)]L_2 \qquad (13b)$$

where $\bar{w}_{l2}$ is the average downwash produced on the tail by wings trailing vortex system so it varies from $\epsilon_{l\infty}(\sigma b_l/b_2)$ as $l\to\infty$ to $\frac{1}{2}\epsilon_{l\infty}(\sigma b_l/b_2)$ as $l\to 0$. Therefore, a positive tail upload increases the drag induced on the tail while the wing's induced drag is decreased by the same amount so as to satisfy Eq. (12). In addition, the bound circulation vortex of the wing has the same effect, which is easily seen in the two-

Table 1 Variation of tail lift and minimum induced drag with gap and span ratios

b_2/b_l	g/b_l	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	∞
0.3	σ	0.3	0.264	0.236	0.210	0.186	0.164	0.144	0.126	0
	L_2/L_l	0	0.01173	0.02067	0.02882	0.03640	0.04291	0.04891	0.05425	0.09
	D_m/D_0	1	0.9986	0.9957	0.9916	0.9867	0.9813	0.9758	0.9702	0.9174
0.6	σ	0.6	0.525	0.460	0.404	0.354	0.310	0.272	0.238	0
	L_2/L_l	0	0.06569	0.1160	0.1552	0.1874	0.2138	0.2352	0.2534	0.36
	D_m/D_0	1	0.9923	0.9757	0.9561	0.9353	0.9149	0.8959	0.8780	0.7353
0.8	σ	0.8	0.681	0.582	0.502	0.433	0.375	0.328	0.290	0
	L_2/L_l	0	0.2091	0.3263	0.3984	0.4492	0.4857	0.5119	0.5313	0.64
	D_m/D_0	1	0.9743	0.9330	0.8939	0.8578	0.8263	0.8002	0.7788	0.6098
1	σ	1	0.780	0.655	0.559	0.484	0.421	0.369	0.326	0
	L_2/L_l	0 to 1	1	1	1	1	1	1	1	1
	D_m/D_0	1	0.890	0.8275	0.7795	0.7421	0.7105	0.6845	0.6630	0.5

dimensional case when we can express the mutually induced drag forces per unit span dimension as

$$\Delta D_{12} = \frac{L_1 L_2}{4\pi q} \frac{l}{l^2 + g^2} = -\Delta D_{21} \tag{13c}$$

This two-dimensional relation is independent of the relative chord lengths of the wing or the tail, and similarly Eqs. (13a) and (13b) are independent of the tail size as long as $b_2 \leq b_1$. As first shown by Prandtl[4] the complete circulation vortex system of lifting surface 2, which we will consider to be the tail, will rotate the lift vector of lifting surface 1 so as to produce a thrust component as long as 1 is ahead of 2. If $b_2 \ll b_1$, then this lift vector rotation of 1 is primarily accomplished by the trailing vortex system of 2. This fact has a simple physical explanation when one notes that the trailing vortex system produced by a tail upload induces an upwash over the wing's outer segments ($|y_1| > b_2/2$) while the downwash produced within the tail's span ($|y_1| < b_2/2$) is more than overcome by the upwash produced by the tail's bound circulation vortex. When $b_2 \to b_1$ then it is the tail's bound circulation vortex that primarily rotates the wings lift vector so as to produce a "wing thrust" that cancels the additional drag induced on the tail, in a manner similar to that found in Eq. (13c). It is important to note that although $(D_{12})_\infty$ is negative with a tail download, Eqs. (3) and (10) prove that the increase in the induced drag of the wing when $L_1 > W$ is sufficient to make the total induced drag greater with a tail download than it would be for a tail upload of the same magnitude, even though $(D_{12})_\infty$ would then be positive, as long as the gap is finite so $(\sigma b_1/b_2) < 1$.

Prandtl[3] and Glauert[5] both used the zero stagger position to evaluate σ, however, for a small tail span ($b_2 \ll b_1$) the Trefftz-plane concept, corresponding to an infinite stagger distance, leads to a much simpler evaluation of σ because the Trefftz-plane downwash of the wing is nearly uniform over a small span tail as shown in Prandtl's[3] Fig. 3 (which is duplicated as Fig. 92 on p. 164 of Glauert,[5] and Fig. 65 on p. 151 of Durand[6]). Following these references, we rewrite Eq. (12) as

$$(D_{12})_\infty = \frac{\sigma b_1}{b_2} \epsilon_{1\infty} L_2 = \int_{-b_2/2}^{b_2/2} \frac{w_{12}(\infty, y_2, g)}{v} \mathrm{d}L_2(y_2) \tag{14}$$

When the gap is zero then the downwash is the constant value $\epsilon_{1\infty} = 2C_{L_1}/\pi A_1$ over the entire tail span so that $(\sigma b_1/b_2) = 1$ for any $b_2 \leq b_1$. Then for any gap we have

$$\frac{w_{12}(\infty,0,g)}{v} = \epsilon_{1\infty}\left[1 - \tanh\left(\sinh^{-1}\frac{2g}{b_1}\right)\right] > \frac{w_{12}(\infty, b_2/2, g)}{v}$$

from Glauert[5] (p. 164) or Durand[6] [p. 149, Eq. (26.14)]. The spanwise distribution of w_{12} referred to shows that it is approximately constant over the tail span as long as $b_2/b_1 < 0.3$; therefore, we may evaluate σ from Eq. (14) as

$$\frac{\sigma b_1}{b_2} \approx \left[1 - \tanh\left(\sinh^{-1}\frac{2g}{b_1}\right)\right] = 1 - \frac{2g}{b_1}\left[1 + \left(\frac{2g}{b_1}\right)^2\right]^{-1/2} \tag{15}$$

This approximation for σ turns out to be surprisingly accurate, giving the values for $b_2/b_1 = 0.6$ within 7%, the σ values for $b_2/b_1 = 0.3$ within 2%, and having less than 1% error for all $b_2/b_1 \leq 1/4$. This approximation for σ was used to prepare Figs. 1 and 2 for values of $b_2/b_1 < 0.3$. These graphs are more useful for analyzing wing-tail combinations than the graph presented by Prandtl,[3] and duplicated in Ref. 6 (p. 221, Fig. 88b), which is applicable to biplanes of nearly equal span. For $b_2/b_1 \geq 0.6$ the numerical values for Figs. 1 and 2, as well as those for D_m/D_0 in Table 1, may be obtained directly from Prandtl's Table II on p. 13 of Ref. 3. The agreement is within 1%, which is remarkable when one notes that Prandtl's values were obtained through the mechanical integration of hand drawn graphs by a planimeter. Prandtl's numerical values were duplicated on p. 220 of Ref. 6, except the first row heading is incorrectly labeled (gap/avg. span) instead of (g/b_1) as originally given by Prandtl.[3]

III. Increase of Induced Drag in a Real Viscous Flow

As shown by Glauert's calculations of the downwash behind the centerline of the wing (Fig. 94 on p. 169 of Ref. 5), the nearer the tail is to the wing the greater the error in overestimating the tail thrust if one uses ϵ_W rather than the proper value of $\epsilon_{1\infty}$. In addition Glauert's downwash calculations show that $\epsilon_{1\infty}$ can be considerably decreased by any departure from the ideal potential flow elliptic loading. For example the rollup of the trailing circulation vortex sheet of an elliptic wing loading reduces $\epsilon_{1\infty}$ from $2C_{L_1}/\pi A_1$ to $1.62\, C_{L_1}/\pi A_1$, a decrease of 19%. Also, all real viscosity and aerodynamic interference effects increase the induced drag produced by each lifting surface, consequently Eq. (9) could be rewritten for real viscous flow as

$$C_{D_i} = \frac{C_{L1}^2}{\pi A_1 e_1} + \frac{\sigma b_1}{b_2 e_3} \frac{2C_{L1}}{\pi A_1} EC_{L2} + \frac{S_1}{S_2} \frac{(EC_{L2})^2}{\pi A_2 e_2} \tag{16}$$

where $e_2 \leq e_1 < 1 < e_3$, noting $e_3 = 1.23$ for the inviscid rollup of the wing's elliptic loading with respect to a small span tail behind the wing's centerline. Now we can eliminate C_{L_1} by introducing Eq. (7) into Eq. (16) and utilizing $C_{L_1} = C_{L0} - EC_{L_2}$ to obtain

$$C_{D_i} = \frac{C_{L0}^2}{\pi A_1 e_1} - \left(1 - \frac{\sigma b_1 e_1}{b_2 e_3}\right) \frac{2C_{L0}}{\pi A_1 e_1} EC_{L_2} + \left(1 - \frac{2\sigma b_1 e_1}{b_2 e_3} + \frac{b_1^2 e_1}{b_2^2 e_2}\right) \frac{(EC_{L2})^2}{\pi A_1 e_1} \tag{17}$$

The middle term now proves that the induced drag is always greater with a tail download since $\sigma b_1/b_2 \leq 1$ and $e_1 < 1 < e_3$. The minimum induced drag is given by the positive tail load defined by

$$\frac{EC_{L2}}{C_{L0}} = \left(1 - \frac{\sigma b_1 e_1}{b_2 e_3}\right)\left(1 - \frac{2\sigma b_1 e_1}{b_2 e_3} + \frac{b_1^2 e_1}{b_2^2 e_2}\right)^{-1} = \frac{L_2}{W} \tag{18}$$

Therefore Eq. (17) gives the minimum induced drag as

$$C_{D_m} = \frac{C_{L0}^2}{\pi A_1 e_1}\left(1 - \frac{\sigma^2 e_1 e_2}{e_3^2}\right)\left[1 + \frac{b_2^2 e_2}{b_1^2 e_1}\left(1 - \frac{2\sigma b_1 e_1}{b_2 e_3}\right)\right]^{-1} \tag{19}$$

This proves that $C_{D_m} < C_{L0}^2/\pi A_1 e_1$, (the induced drag with zero tail load). Equation (19) reduces to Eq. (6) when all $e_n = 1$, however because $e_2 \leq e_1 < 1 < e_3$, it is evident that in a real viscous flow the induced drag can now be reduced by an even more positive tail load than that required in potential flow.

The last term in Eq. (17) is written in a form that most clearly emphasizes the advantages of an efficient large aspect ratio wing. This term can also be expressed in a more familiar form, corresponding to Eq. (10), as

$$\left[1 + \frac{b_2^2 e_2}{b_1^2 e_1}\left(1 - \frac{2\sigma b_1 e_1}{b_2 e_3}\right)\right]\left(\frac{q_2}{q}\right)^2 \frac{S_2}{S_1} \frac{C_{L2}^2}{\pi A_2 e_2} \tag{20}$$

Note that this contains $(q_2/q)^2$, the same as the last terms in both Eqs. (9) and (16). This is contrary to present day usage (e.g., see Refs. 7 and 8 and the texts they refer to) which assumes only q_2/q in this term. However, Eqs. (1) and (9) definitely show that this last term, which is for the induced drag of the tail itself, should contain $(q_2/q)^2 = \eta^2$.

In Refs. 7–9 the term ϵ_W in Eq. (11) is replaced by $\epsilon_0 + \epsilon_\alpha \alpha$, where ϵ_0 is the total downwash on the tail when $L_I = 0$, including the combined effects of the wing and fuselage. The calculations in Ref. 8 obviously show that if ϵ_0 is sufficiently positive, then the minimum total drag will occur with a tail download. However, these calculations neglect the downwash on the wing that is produced by the download on the tail, and if this effect is included it should result in a minimum induced drag at a positive tail load. It is important to note that in a uniform potential inviscid flow the downwash ϵ_0 from any part of the aircraft should not produce a further decrease in the total induced drag with a tail download, since any downwash ϵ_0 would increase the tail download and therefore increase the tail's downwash acting on the wing. This should create a wing lift and induced drag component that could exactly cancel the additional "tail thrust." Only an experimental investigation could determine if any viscous flow effects could possibly increase the "tail thrust" by any ϵ_0. In addition, the effect of the Trefftz-plane two-dimensional downwash produced by the fuselage should be investigated because this could possibly increase the total induced drag with a tail upload. In any case, the minimum total induced drag should occur with $L_2 \approx 0$. However, the minimum total drag could occur with a tail download if the tail incidence angle was so negative that at small C_{L0} the total aircraft drag was a minimum at an angle of attack corresponding to a tail download. This could occur with a nearly zero tail gap because, as shown in Fig. 3, the minimum induced drag is given by a zero tail load, and the induced drag is the same with either a tail upload or download, see Eq. (2) and the calculations given by Larabee.[10] However, a positive tail load is required if the tail is not in the plane of the wing, as illustrated in Fig. 3. It should also be noted that g/b_I is actually increased for a high tail because the trailing circulation vortices move downward behind the wing. Conversely, an increase in the angle of attack would decrease g/b_I for a high tail, and increase it for a low tail. The calculations in Ref. 7 neglect the upwash produced on the wing by the tail upload, and the wind tunnel tests (Fig. 4 on p. 625 of Ref. 7) give less drag than calculated. This decrease in drag is ascribed to a possible variation in the wing-fuselage drag. However, part of this decrease in drag must be due to the forward rotation of the wing's lift vector because of the upwash produced by the tail upload, as previously discussed. A careful scrutiny of published wind tunnel data shows that the total drag with a tail download is usually greater than that with a corresponding tail upload, as long as $C_{L0} > 0.3$ so that the induced drag is over 20% of the total drag. This observation is in agreement with the middle term of Eq. (17) which states that the induced drag with a tail download is greater than that for the same magnitude upload by the amount given by

$$\Delta C_{D_i} = 2(1 - \sigma b_I e_I / b_2 e_3)(2C_{L0}/\pi A_I e_I) E |C_{L_2}| \tag{21}$$

For the numerical values given in Fig. 3 for $g/b_I = 0.2$ ($\sigma = 0.186$) we obtain $\Delta C_{D_i} = 0.0304\ EC_{L_2}$. However, the values in Fig. 3 are for ideal potential flow with all $e_n = 1$. If we introduce the reasonable values $e_I = 0.95$, $e_2 = 0.90$, $e_3 = 1.23$ (corresponding to the rollup of the wing's vortex sheet) into Eq. (17) for this same aircraft we now find that a tail download of $L_2/W = -0.05$ produces $C_{D_i} = 0.01002$ which is 8.2% greater than $C_{D0} = 0.00926$ for a zero tail load, and 10.7% greater than $C_{D_m} = 0.00905$ for a tail upload of $EC_{L_2} = 0.0195$ ($L_2/W = 0.0443$). Consequently, a tail download that is 5% of the aircraft's weight has increased the induced drag by 8.2% over that for a zero tail load. This is a typical tail download that is carried by modern transport aircraft, and if the effective zero lift drag of the aircraft is represented by $C_{D_e} = 0.013$ then this 5% W tail download would increase the total drag by 3.4% over the zero tail load trim, and 4.4% over the total drag with the optimum tail upload. It must be noted that at high subsonic speeds the compressibility drag of the wing would be severely increased by a tail download since $L_I > W$.

IV. Location of c.g. for Zero Tail Load

The relation between the tail-off neutral point h_0, the tail-off pitching moment C_{m0} about h_0, and the tail lift required for trim about any c.g. location may be written as

$$\frac{\text{c.g.}}{C_I} = h_0 + \left(\frac{l}{C_I} - \frac{\text{c.g.}}{C_I}\right)\frac{L_2}{W} - \frac{C_{m0}}{C_{L0}} \tag{22}$$

where c.g., h_0, C_I, and the tail length l, are all measured from the leading edge of the wings mean aerodynamic chord (C_I). The c.g. location for minimum induced drag can now be easily obtained by simply substituting L_2/W from either Eq. (4) or Eq. (18) into Eq. (22). However for a typical small tail, Fig. 1 shows that the minimum induced drag (D_m) is practically the same as the induced drag with zero tail load (D_0); consequently the additional aft location of the c.g. when $L_2 > 0$ is not very beneficial, e.g., if $b_2/b_I = 0.3$ and $g/b_I = 0.2$ then Table 1 shows that D_m is only 1.33% less than D_0 when $L_2/L_I = 0.0362$, therefore if $l/C_I = 3$ and $h_0 = ¼$ we find that the c.g. must be moved aft an additional distance equal to 0.096 C_I in order to realize a 1.33% reduction in the total induced drag. If the gap ratio is only 1/10 of the wing span than the decrease in drag is only 0.43% from that with zero tail load. Consequently for most wing-tail combinations the simplest procedure is to try and achieve $L_2 \approx 0$, as shown by Naylor[1] and Larrabee.[10] This would require a sufficiently large "tail volume" in order to attain the necessary aft c.g. and still maintain static longitudinal stability as defined by

$$\frac{dC_{m\,\text{c.g.}}}{dC_{L_I}} = \left[\frac{\text{c.g.}}{C_I} - h_0 - \frac{E}{C_I}(l - \text{c.g.})\frac{1+2/A_I}{1+2/A_2} \times \left(1 - \frac{d\epsilon_W}{d\alpha}\right) + \left(\frac{\Delta M\alpha}{qS_I C_I}\right)\Big/\left(\frac{dC_{L_I}}{d\alpha}\right)\right] < 0 \tag{23}$$

Since the tail length l is measured from the wing leading edge, rather than from the c.g. as in Ref. 1, the tail volume contains the term ($l -$c.g.). The additional term $\Delta M\alpha$ is primarily the unstable moment produced by the fuselage, whose order of magnitude may be roughly estimated by $\Delta M\alpha \approx 2q$ (fuselage volume) in order to ascertain if it must be considered. Also, if the tail length $l > b/2$ then $d\epsilon_W/d\alpha \approx d\epsilon_{I\infty}/d\alpha = 4/(2 + A_I)$ in agreement with the relations used by Naylor.[1] Then for $L_2 = 0$ (so $C_{L_I} = C_{L0}$) we simplify Eqs. (22) and (23) to

$$\frac{dC_{m\,\text{c.g.}}}{dC_{L0}} \approx \left[-\frac{C_{m0}}{C_{L0}} - E\left(\frac{l}{C_I} - h_0 + \frac{C_{m0}}{C_{L0}}\right)\frac{1+2/A_I}{1+2/A_2} \times \frac{A_I - 2}{A_I + 2} + \frac{2(\text{fuselage vol.})}{S_I C_I}\frac{1+2/A_I}{2\pi}\right] < 0 \tag{24}$$

Now if we use the data for the aircraft depicted in Fig. 3, and in addition assume $h_0 \approx ¼$, and neglect the fuselage unstable moment, we obtain

$$\frac{dC_{m\,\text{c.g.}}}{dC_{L0}} \approx -1.087\frac{C_{m0}}{C_{L0}} - 0.0874\left(\frac{l}{C_I} - \frac{1}{4}\right)$$

$$(\text{c.g.}/C_I) \approx ¼ - C_{m0}/C_{L0}$$

e.g., if $C_{L0} = 0.44$ and $C_{m0} = -0.088$ we find that $(\text{c.g.}/C_I) = 0.45$ and $l/C_I = 2.74$ for neutral stability, while $l/C_I = 3$ gives $dC_m/dC_L = -0.023$, showing that the latter tail length would give sufficient longitudinal stability with zero tail load for the aircraft considered in Fig. 3 when $C_{m0} \geq -0.088$.

References

[1] Naylor, C.H., "Notes on the Induced Drag of Wing-Tail Combination," British R&M 2528, July 1946.

[2] Laitone, E.V., "Ideal Tail Load for Minimum Aircraft Drag," *Journal of Aircraft*, Vol. 15, March 1978, pp. 190 – 192.

[3] Prandtl, L., "Induced Drag of Multiplanes," NACA TN 182, March 1924.

[4] Prandtl, L., "Applications of Modern Hydrodynamics to Aeronautics," NACA TR 116, 1921.

[5] Glauert, H., *The Elements of Aerofoil and Airscrew Theory*, 2nd Ed., Cambridge University Press, 1947, pp. 162 – 169 and 181 – 184.

[6] Durand, W.F., *Aerodynamic Theory*, Vol. 2, Springer, Berlin, 1934, pp. 146 – 151 and 216 – 221.

[7] McKinney, L.W. and Dollyhigh, S.M., "Some Trim Drag Considerations for Maneuvering Aircraft," *Journal of Aircraft*, Vol. 8, Aug. 1971, pp. 623 – 629.

[8] Lutze, F.H., "Trimmed Drag Considerations," *Journal of Aircraft*, Vol. 14, June 1977, pp. 544 – 546.

[9] Hoerner, S.F., *Fluid-Dynamic Drag*, 2nd Ed., published by the author, New York City, 1958, Chap. 7, p. 15.

[10] Larrabee, E.E., "Trim Drag in the Light of Munk's Stagger Theorem," *Proceedings of the NASA Industry, University General Aviation Drag Reduction Workshop*, edited by J. Roskam, NASA CR 145627, July 14 – 16, 1975, pp. 319 – 328.

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